MATH 504 HOMEWORK 7

Due Monday, November 5.

Problem 1. *(The Δ-system lemma) Suppose that D is a set, and $\{d_\eta \mid \eta < \omega_1\}$ is a family of distinct finite subsets of D. Show that there is an uncountable $I \subset \omega_1$ such that $\{d_\eta \mid \eta \in I\}$ forms a Δ-system, i.e. for some kernel d, for all $\eta < \xi$ both in I, $d_\eta \cap d_\xi = d$.*

Problem 2. *Let M be a transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that $p \in \mathbb{P}$ is such that $p \Vdash$ "$\dot{f} : \lambda \to \tau$ is a function".*

1. *Show that for every $\alpha < \lambda$, $\{q \mid \exists \gamma \in \tau (q \Vdash \dot{f}(\alpha) = \gamma)\}$ is dense below p.*
2. *Let $B = \{\gamma < \tau \mid (\exists q \leq p)(\exists \alpha < \lambda)(q \Vdash \dot{f}(\alpha) = \gamma)\}$. Show that if $\sup(B) < \tau$, then $p \Vdash$ "$\operatorname{ran}(\dot{f})$ is bounded".*

Problem 3. *Prove (in detail) that if $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.*

Problem 4. *Suppose that $\mathbb{P}$ is a poset, $A \subset \mathbb{P}$ is a maximal antichain, $\phi(x)$ is a formula, and $\langle \tau_p \mid p \in A \rangle$ are $\mathbb{P}$ names such that for all $p \in D$, $p \Vdash \phi(\tau_p)$. Show that there is a $\mathbb{P}$ name τ, such that $1_{\mathbb{P}} \Vdash \phi(\tau)$.*

We say that σ is a *nice name* for a subset of ω, if for all $n \in \operatorname{dom}(\sigma)$, $\{q \mid \langle n, q \rangle \in \sigma\}$ is an antichain.

Problem 5. *Suppose that $M \models \kappa^\omega = \kappa$, and let G be $Add(\omega, \kappa)$-generic over M. Show that $M[G] \models 2^\omega = \kappa$.*

Hint: we already showed that $M[G] \models 2^\omega \geq \kappa$. For the other direction, argue that it is enough to consider the possible number of nice names for subsets of ω. I.e. for any $a \subset \omega$ in $M[G]$, show there is a nice name $\dot{a}$ such that $\dot{a}_G = a$. Then use the c.c.c. to compute the number of nice names.

Note that in view of the last problem, we can make 2^ω be any κ with $\operatorname{cf}(\kappa) > \omega$. This is optimal, since by a corollary of Konig's lemma we know that $\operatorname{cf}(2^\omega) > \omega$.